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UNITED STATES NAVAL POSTGRADUATE SCHOOL

A MODEL OF ADMINISTRATIVE CHOICE

IN THE SYSTEMS ANALYSIS ENVIRONMENT

by

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ABSTRACT:

In this paper a mathematical model of governmental decision-maker's problem of choice from a set of alternative cost-benefit streams is presented. The choice objects, costs, and benefits are considered to be defined physically/socially, in time, in space, and by a state-of-nature. The decision-maker's preferences with respect to costs and benefits are represented by a utility index of the standard type except that costs are assumed to be a disutility-causing entity. The set of alternatives from which choice is described is represented by a cost-benefit surface. The decision problem is formulated as a maximization problem. Decision rules are derived. Comparative statics results are given.

Prepared by:

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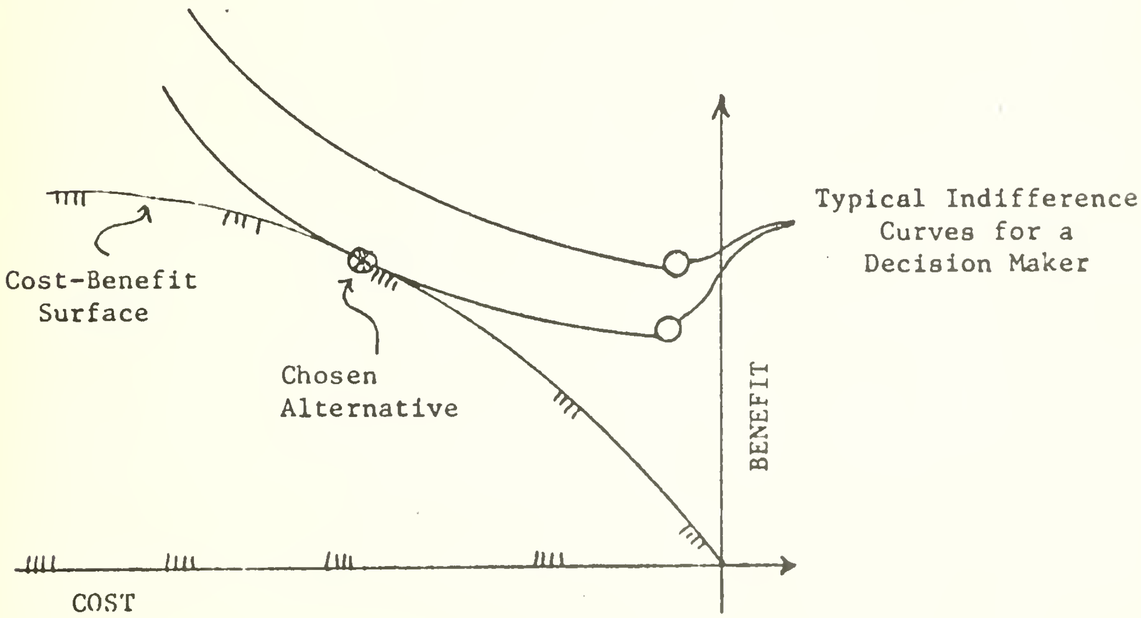
SECTION I

INTRODUCTION

This paper is concerned with a description of a governmental decision-maker choosing among alternatives whose costs and benefits have been illuminated analytically. The decision-maker is considered to be involved in a planning, programming and budgeting system and to be responsible for at least some area where cost-benefit studies can be helpful. The decision maker's study team is envisaged as being given an assignment to develop the alternatives and their costs and benefits. The output of the study team is some representation of a cost-benefit surface.



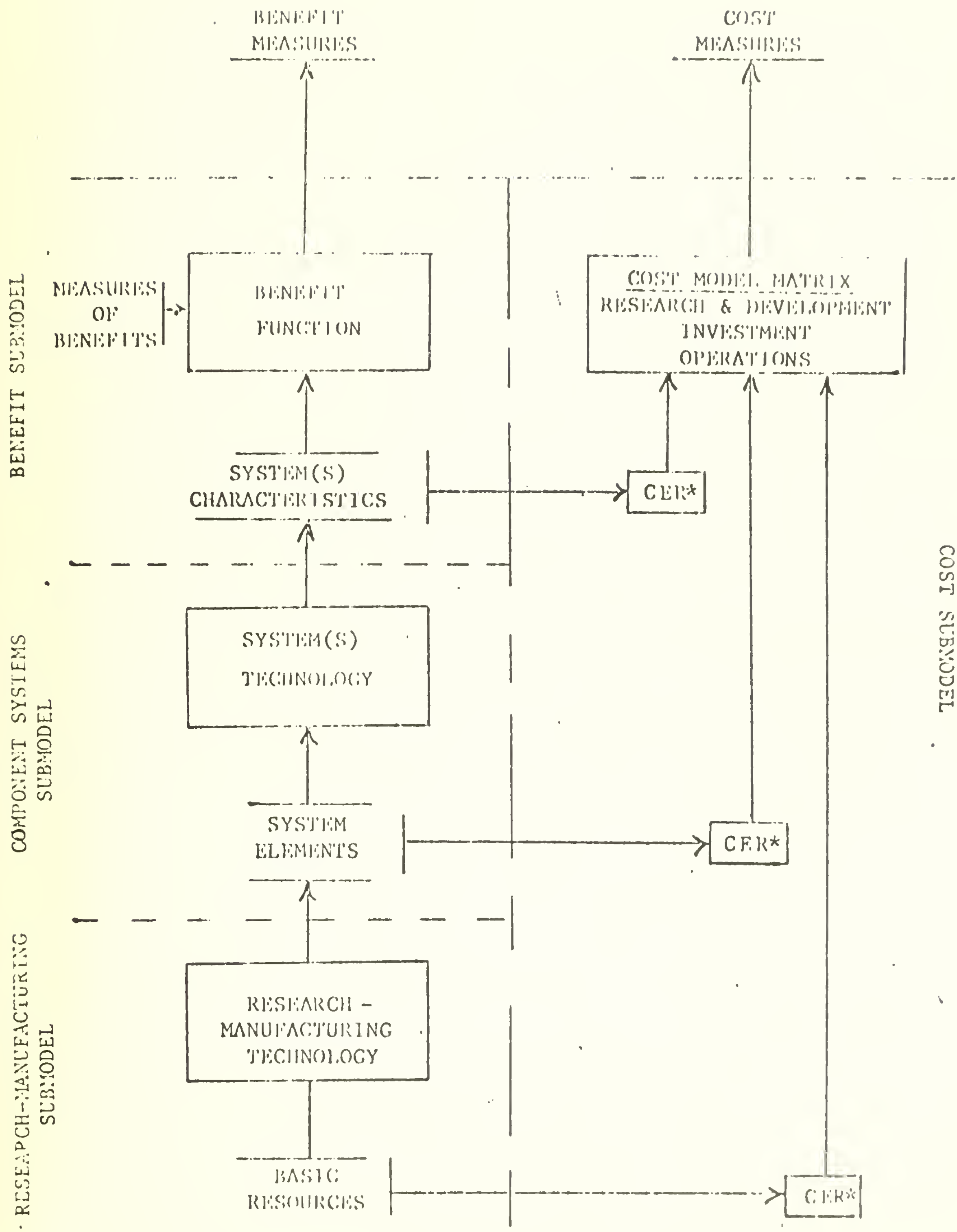
SIMPLIFIED GEOMETRY OF THE MODEL

FIGURE 1

The phenomenon to be described, then, is a governmental decision-maker, his study team, and their interaction. In a very simplified fashion Figure 1 shows a cost-benefit surface developed by a study team. A decision-maker's indifference curves are also shown, as is the subjectively selected optimum alternative. The details of the study team's deliverations are not shown by this figure. They are shown schematically in Figure 2.

As can be seen in the schematic, costs and benefits are produced by the study team as an interrelated flow among benefit, component systems, research-manufacturing, and cost submodels. The details of the operational definitions of the variables will be given in Section II. As can be seen by studying the schematic, basic resources (e.g., engineering hours, raw materials, tooling) are transformed into system elements (e.g., in the military context, tanks, planes, trained personnel). These system elements are the inputs to the component systems submodel. The outputs of this submodel are the system characteristics (e.g., in transportation, range, payload, speed, fuel consumption). These characteristics are produced from the system elements. Finally, characteristics are transformed into values of the system benefit measures (e.g., in poverty programs, expected income distributions).

The inputs to the cost model are characteristics, elements, and resources. By use of the cost estimating relationships, the cost model matrix can be computed and the cost measure(s) obtained. All these input types are considered to allow for such phenomenon as learning curves, quantity discounts, and rather detailed disaggregated



*Cost Estimating Relationship

SCHEMATIC OF THE STUDY TEAM'S DELIBERATIONS

FIGURE 2

estimation. The cost model matrix has columns for the time periods of the analysis and rows for the system elements. The columns can be grouped by research and development costs, investment costs, and operating costs, if this is desirable. Some elements of the matrix may, of course, be zero. The cost measure values are computed by pre- and post-multiplication of the matrix by appropriate vectors. For example, if present costs are to be computed, then the pre-multiplication is by a sum vector and the post-multiplication is by a vector of discount factors.

Such a disaggregated model can be used to study the surface relating the various cost and benefit measures. Analogous to production theory in economics, this is a production possibility surface where each point is vectorially undominated. The surface can be considered as some function of all cost and benefit measures equal to zero -- an implicit function. The implicit function is interpreted with benefits as outputs and costs as inputs and is the surface diagrammed in Figure 1.

The literature of mathematical models of such a phenomenon is small. Heuston and Ogawa [1] and the references there are the appropriate ones. This paper attempts to broaden the mathematical framework for describing the phenomenon of cost-benefit alternative choice. As such, it is somewhat a synthesis of the previous papers and also a generalization in that the previous models can be considered special cases of the model presented here. Another major difference is the stress here on the incommensurability of costs and benefits in many problems. As such, commensurability can be considered as net benefit measures, which, of course, are very desirable when available.

SECTION II

THE CHOICE OBJECTS

As discussed in the previous section, the decision-maker is modeled as choosing from a set of cost-benefit vectors that is generated by an analysis team. In this section the choice objects of the decision-maker are given operational meaning. The choice objects are the benefit measures and the cost measures. These variables are assumed to have physical-social, time, space, and state-of-nature attributes. In addition to these variables, the exogenous variables where they are discussed in later sections also are assumed to have these attributes. The attributes will be discussed in turn.

The physical attributes of a measure have been discussed before [1]. It is stressed, though, that the same physical and/or social phenomenon can be measured in multiple ways -- and they can all be important. For example, Miller, et al., [2] have listed the physical-social (my terminology) measures of poverty as income (threshold, relative, share of national income), assets (housing, consumer durables, savings, insurance), and services (education, health, neighborhood amenities, protection, social services, transportation). In considering this model, the reader is urged to regard some of the multiple measures as being associated with the same physical/social phenomenon.

The second attribute is time dating. With this attribute, the same physical/social measure at two different dates will be treated as two different measures. In this fashion, choice object time streams can be associated with a project. It is noted that the time attribute is associated with such measures as present cost and present benefits, since while they are calculated with many dates, they are calculated as of some particular date.

The third attribute locates the measure of the phenomenon in physical space. Hence, the same physical-social measure at two different locations will be treated as two different measures. A location is determined by categorizing the spatial extension of the phenomenon into elementary regions.

The risk or state-of-nature attribute will be modeled in the Debreusian manner [3]. That is, the future will be modeled as a time sequence of states-of-nature. At any one date, the states-of-nature are assertions concerning all that can conceivably happen including natural phenomenon, technological change, political acts, and the like. It is usual to model this as an event tree [4].

In cost-benefit analysis, particularly as used in the defense department, the scenario has been an important tool. A scenario seems to have no concise definition. However, it is used to mean the background aspects of a given situation. Here, scenario will be used to denote a unicursal path (a path with no steps retraced) through the event tree. Hence, in a model with only two dates (present and future), scenario and state-of-nature become synonymous. In summary, then, choice variables are defined to have an attribute for the state-of-nature that could prevail at a given date.

The above concept of state-of-nature is extended here to include the empirical relevance of alternative methods and models. As most practitioners have undoubtedly noticed during a study, discussion concerning the empirical relevance -- "realism" -- of alternative methods and models is often heated and lengthy. It is clear that such disagreement could be resolved by appropriate experimentation and application of scientific procedures. However, since the time frame of the decision does not always allow such experimentation and since the resources for such experimentation may not be available, an attribute of empirical relevance is included in the concept of state-of-nature.

The choice objects, defined with physical-social, time, space, and risk attributes, must also be scaled and given mathematical structure. Here, the details of the scaling will not be considered [5]. Rather, each measure is assumed to have an associated multiplicative scale. This scale is represented by the real numbers.

SECTION III

THE DECISION-MAKER'S UTILITY FUNCTION

As is usual in such models, the decision maker is assumed to have a complete preference preordering with respect to the cost-benefit vectors. In addition, continuous and strongly convex preferences are assumed. These assumptions assure that an ordinal scale utility index, U, can be constructed [3]. Again, as is usual, the utility index is assumed to have continuous second partial derivatives. Certain results also require the utility index to be concave, but this is not assumed throughout the paper.

Unlike standard consumer theory in economics, all the choice objects are not "utility producing" objects. Rather, costs are "disutility producing" objects. Hence, the marginal utility[1] of costs is negative. To retain the positive marginal utilities, it is assumed that costs are measured with a negative sign. In the two-dimensional case, Figure 4 is the typical representation of indifference curves or isoutility lines.

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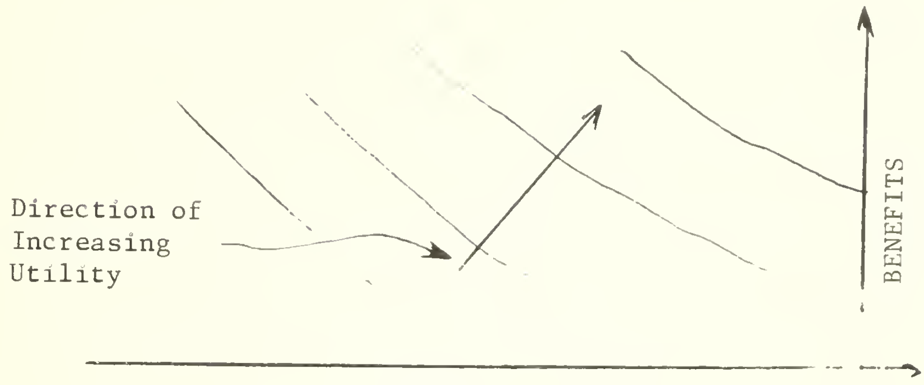
[1] Since the utility index is known only up to a monotonic increasing transformation, marginal utility here means the rate of change of the transformed index with respect to the original (always positive) times the original marginal utility.

As is well known, the slope of an indifference curve measures the "trade-off" internal to the individual between the choice objects. In



Cost-Benefit Indifference Curves

FIGURE 4

the usual manner[1] the negative of the slope of an indifference curve will be defined as the rate of psychological xxxxxxxxxxxx substitution. The x's are replaced by an appropriate modifier, as noted in the next few sentences. When the "trade-off" is between cost m and benefit ℓ, the name will be the rate of psychological cost m-benefit ℓ substitution.[2] This will be denoted as $RPC_mE_\ell S$. This measures an individual decision-maker's willingness to gain additional effectiveness at an additional cost. When two cost measures are considered, the name becomes rate of psychological cost measures substitution. This will be denoted $RPC_mC_\alpha S$. It measures the

[1] It is noted that many names exist for this concept. The present one was chosen to emphasize the subjective nature of this "trade-off."

[2] E denotes benefits and C costs with the measure listed first considered the dependent variable and the other the independent variable. Since indifference curves are one-to-one correspondences, the permuted rate of substitution is the inverse of the one defined. The same convention will hold for isoquandt analyses.

decision-maker's willingness to substitute between two different measures of resource use. Finally, when two benefit measures are under consideration, the words are rate of psychological benefit measure substitution. This is denoted RPE_jE_kS. It measures the willingness to substitute between two measures of benefits.

As the word psychological connotes, choice by the decision-maker is internal. This appears to be what some call "judgment." In this paper "judgment" will be dichotomized into ethics or empirical relevance categories. The first category includes such considerations as what should be the role of government, the size of government, the "worth" of the citizens benefited by the project, and the "amount" of consideration given to future generations. The reader can undoubtedly supply many more. The second category includes such items as the reliability of forecasting techniques, the relevance of specific cost and benefit measures to the "real phenomenon," the degree of approximation exhibited by different models, and the degree to which "immeasurables" are "measured." Again, the reader can undoubtedly supply many more. This last category includes all matters of judgment which could be resolved by application of appropriate scientific procedures. Presumably, in a world with costless information this last category would not be needed. While it is important to distinguish these categories, the decision-maker's preferences seem to "mix them all up," and hence only one inseparable utility index (and any monotonic increasing transformation thereof) will be formulated.

SECTION IV

THE COST-BENEFIT FUNCTION

The implicit function form of the cost-benefit function will be used ($H(\underline{E}, \underline{C}, \underline{r}) = 0$). To further illuminate its interpretation, it is noted that some of the measures can be cost-benefit ratios, net benefits, and rate(s) of return of benefit over cost. Clearly many measures, particularly those just indicated, require costs and benefits to be commensurable. In many instances this commensurability does not apply. When commensurability is considered, it seems natural to permit at least some benefit measures to be unrestricted in sign. This is more general and presumably a better description model but greatly complicates the exposition. Hence, since no essential notions are lost, benefit will be treated as nonnegative and costs nonpositive in what follows. The cost measures may be further restricted by the existence of budget limitations ($\underline{B}=(B_1,\ldots,B_m,\ldots,B_M)$).

SECTION V

THE DECISION PROBLEM

In the previous sections, a governmental decision-maker's problem is discussed in terms of its component parts: the choice objects, the utility function, and the cost-benefit function. Here, these components are interrelated by framing the decision-maker's problem as maximizing his utility subject to the cost-benefit function.

In formal terms the decision problem is:

$$\underset{E_1, \ldots, E_L,\ c_1, \ldots, c_M}{\text{Max}}\ U(E_1, \ldots, E_\ell, \ldots, E_n;\ c_1, \ldots, c_m, \ldots, c_M)$$

subject to

$$H(\underline{E}, \underline{c}, \underline{r}) = 0$$

$$c_1 \geq B_1^*$$

$$\vdots$$

$$c_M \geq B_M^*$$

$$E_1, \ldots, E_L \geq 0$$

$$c_1, \ldots, c_M \leq 0$$

where the symbols are interpreted as follows:

$U(\ )$: utility function

E_ℓ: ℓ^{th} benefit (or effectiveness) measure

c_m: m^{th} cost (or resource use) measure

$H(\ )$: implicit cost-benefit function

B_i^*: maximum permissible level of the i^{th} cost measure. Note that $B_i^* \leq 0$ for all i. This is frequently called the budget level.

$\underline{r}$: various parameters as cost-estimating relationship coefficients.

While it is possible to analyze the model as just stated, it is more convenient to transform the problem to consider costs and budgets in positive terms. This is accomplished by the following:

$$\begin{array}{ll} C_1 = -c_1 & B_1 = -B_1^* \\ \vdots & \vdots \\ C_m = -c_m & B_m = -B_m^* \\ \vdots & \vdots \\ C_M = -c_M & B_M = -B_M^* \end{array}$$

With this substitution the model becomes:

$$U(\underline{E}, -\underline{C})$$

$$\underline{E}, \underline{C}$$

$$H(\underline{E}, -\underline{C}, \underline{r}) = 0$$

$$\underline{C} \le \underline{B}$$

$$E_\ell, C_m \ge 0 \quad \forall_{\ell,m}$$

So that the Kuhn-Tucker theorem [6] may be applied, the following Lagrangian is formed.

$$\phi(\underline{E}, \underline{C}, \lambda_o, \ldots, \lambda_M) = U(\underline{E}, -\underline{C})$$

$$-\lambda_o H(\underline{E}, -\underline{C}) - \sum_{i=1}^{M} \lambda_i (C_i - B_i)$$

The necessary conditions for a maximum are:

(1) $\frac{\partial U}{\partial E_\ell} - \lambda_o \frac{\partial H}{\partial E_\ell} \le 0$ for $\ell = 1, 2, \ldots, L$

(2) $\frac{\partial U}{\partial c_m} + \lambda_o \frac{\partial H}{\partial c_m} - \lambda_m \le 0$ for $i = 1, \ldots, M$

(3) $H(\underline{E}, -\underline{C}, \underline{r}) = 0$

(4) $C_m - B_m \le 0$ for $m = 1, \ldots, M$

(5) $E_\ell, C_m \ge 0$ for all ℓ, m

And, also, the following theorems are of interest:

(6) If $C_m \le B_m$, then $\lambda_m = 0$

(7) If $\frac{\partial U}{\partial E_\ell} - \lambda_o \frac{\partial H}{\partial E_\ell} < 0$, then $E_\ell = 0$

(8) If $\frac{\partial U}{\partial c_m} + \lambda_o \frac{\partial H}{\partial c_m} - \lambda_m < 0$, then $C_m = 0$

From these conditions, various decision rules will be derived and interpreted. This will be done first for two benefit measures, secondly for two cost measures, and lastly for one benefit and one

cost measure. The latter two will be considered when budgets are binding and not binding. Finally, it is noted that if the utility function is concave and the production function convex, then the above conditions ((1) - (5)) are also sufficient. Thus, the decision-maker's rules given next will also be sufficient under these circumstances.

Decision Rule for Two Benefit Measures

Since the budget constraints do not affect the decision rule for two benefit measures, they can temporarily be neglected. However, it is possible that one or more of the inequalities in (1) are strict inequalities. Thus, there are cases where (a) both benefit measures are at a positive level and the associated relations in (1) are equalities; (b) one benefit measure is at a positive level, the other at a zero level, and the associated relations in (1) are an equality and strict inequality, respectively; and (c) both benefit measures are at a zero level and strict inequalities hold in both associated relations in (1). The case where the benefit measure is at a zero level and the associated relation in (1) is an equality is not considered here. This is not meant to imply the unimportance of this "tangency" case, only that the essentials are brought out with the other cases.

When equality relationships in (1) are considered with both associated benefit measures at a positive level, the circumstances are similar to the "interior solution" so familiar to the student of economic theory. The division of one of the equations by the other yields:

$$\frac{\frac{\partial U}{\partial E_\ell}}{\frac{\partial U}{\partial E_\alpha}} = \frac{\frac{\partial H}{\partial E_\ell}}{\frac{\partial H}{\partial E_\alpha}} \qquad \alpha \neq \ell$$

Using the terminology developed in previous sections, the decision rule is for the decision-maker to equate the rate of psychological benefit measure substitution to the rate of benefit measure transformation. This decision rule in conjunction with the others will be necessary (and sufficient) depending on the assumptions concerning the utility function and production function as discussed above.

In case (b) where there is a strict inequality and an equality, the decision rule is:

$$\frac{\frac{\partial U}{\partial E_\ell}}{\frac{\partial U}{\partial E_\alpha}} > \frac{\frac{\partial H}{\partial E_\ell}}{\frac{\partial H}{\partial E_\alpha}} \qquad \alpha \neq \ell$$

This form of the decision rule assumes benefit measure α is at a zero level and benefit measure ℓ at a positive level.

As the reader probably suspects, the inequality form of the decision also applies in case (c). The difference is that with case (c) both benefit measures are at a zero level. The importance of cases (b) and (c) is that since measures are physical/social, time, space, and state-of-nature attributed, a choice at a zero level for some measure appears likely. For example, a decision-maker's attitude toward the likelihood of occurrence of some scenario or the empirical relevance of some model can occasion choice at a zero level.

Decision Rule for Two Cost Measures

Inasmuch as the nonbinding budget constraint cost measure cases are analogous to the above benefit measure cases, these will be considered first. The binding budget constraint case will be considered thereafter.

In its equality form the decision rule is:

$$\frac{\frac{\partial U}{\partial c_m}}{\frac{\partial U}{\partial c_\alpha}} = \frac{\frac{\partial H}{\partial c_m}}{\frac{\partial H}{\partial c_\alpha}} \qquad \alpha \neq m$$

This rule occurs when both cost measures are at a positive level and the relationships in (2) are equalities (case (a), above). It also occurs if one or both of the cost measures are at a zero level and there is tangency of the indifference curves and production isoquandts on the boundary of the positive orthant. This decision rule requires the decision-maker to equate the rate of psychological cost measure substitution to the rate of cost measure substitution.

The inequality form

$$\frac{\frac{\partial U}{\partial c_m}}{\frac{\partial U}{\partial c_\alpha}} > \frac{\frac{\partial H}{\partial c_m}}{\frac{\partial H}{\partial c_\alpha}} \qquad \alpha \neq m$$

of the decision rule is applicable when cost measure m is at a positive level and cost measure α is at a zero level or both are at a zero level. Again, this is important in that cost measures also have all the attributes discussed for benefit measures and a zero level has an interpretation; for example, with respect to the decision-maker's preference over space and time.

When the budget level is binding, the associated Lagrange multipliers are no longer zero. The decision rule developed here will be for equality relationships in (2) and positive Lagrange multipliers. Choosing two of the equality relationships in (2) and performing a little algebra yields:

$$\frac{\lambda_m}{\lambda_\alpha} = \frac{\left(-\frac{\partial U}{\partial c_m}\right) + \lambda_o \frac{\partial H}{\partial c_m}}{\left(-\frac{\partial U}{\partial c_\alpha}\right) + \lambda_o \frac{\partial H}{\partial c_\alpha}} \qquad \alpha \neq m$$

From equation set (1) it is known that:

$$\lambda_o = \frac{\partial U}{\partial E_\ell}\left(\frac{\partial H}{\partial E_\ell}\right)^{-1} \qquad \text{for all } \ell$$

Substituting and rearranging yields:

$$\frac{\lambda_m}{\lambda_\alpha} = \frac{\dfrac{\frac{\partial H}{\partial c_m}}{\frac{\partial H}{\partial E_\ell}} - \dfrac{\frac{\partial U}{\partial c_m}}{\frac{\partial U}{\partial E_\ell}}}{\dfrac{\frac{\partial H}{\partial c_\alpha}}{\frac{\partial H}{\partial E_\ell}} - \dfrac{\frac{\partial U}{\partial c_\alpha}}{\frac{\partial U}{\partial E_\alpha}}} \qquad \alpha \neq m$$

But it is now recognized that these ratios of partial derivatives are rates of substitution and transformation either internal or external. Thus the above equation can be written.[1]

$$\frac{\lambda_m}{\lambda_\ell} = \frac{(RE_\ell C_m T) - (RPE_\ell C_m S)}{(RE_\ell C_\alpha T) - (RPE_\ell C_\alpha S)}$$

[1]Note that since the Lagrange multipliers are positive, this equality can be rearranged into an inequality with just the internal and external "trade-offs" present.

Since the various Lagrange multipliers can be interpreted as the marginal utility of the associated budget level, the ratio of two multipliers can be interpreted as the rate of psychological substitution between budget level types.

When one cost measure is constrained by a budget level and another not, the decision rule can be written as

$$(RC_m C_\alpha S) - (RPC_m C_\alpha S) = (\lambda_\alpha)\left(\frac{\partial U}{\partial c_m}\right)^{-1}$$

The term on the right side of the equation can be interpreted as the rate of psychological substitution between cost measure m and budget α, the binding budget.

The analysis between a cost measure m and a benefit measure ℓ proceeds in the same manner. The formal results are shown in Table 7 and can be interpreted as in the preceding cases.

SECTION VI

DEMAND FOR COSTS AND BENEFITS

In an analogous manner to the economic theory of consumer or producer choice, the necessary conditions imply that the optimal level of benefits and costs are functions of the parameters of the cost-benefit production function and the various budget levels. For simplicity, the analysis begins with the case where the necessary conditions (1), (2), and (3) are equations, and benefits and costs are all positive (5). Thus, budgets may be neglected.[1]

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[1] For the sufficient conditions to an inequality constrained maximization problem where the functions are only assumed differentiable, see King [7]. As King shows, neglecting slack constraints on the problem introduced no difficulties. In fact, this exposition provides the basis for analyzing the problem here as a sequence of cases.

COST-BENEFIT MEASURE DECISION RULES

TABLE 7

NAME	FORMAL DECISION RULE	$E_\ell, C_m > 0$	$E_\ell = 0, C_m > 0$	$E_\ell > 0, C_m = 0$	$E_\ell, C_m = 0$
BUDGET NOT BINDING	$RPC_mE_\ell S \boxed{?} RC_mE_\ell S$	=	<	>	<
BUDGET BINDING	$RE_\ell C_mT - RPE\ C_mS \boxed{?} \lambda_m \left(\frac{\partial U}{\partial E_\ell}\right)^{-1}$	=	<	>	<

With the foregoing assumptions, the necessary conditions consist of $L + M + 1$ equations (equation sets (1), (2), and (3)) and $L + M + 1$ variables (LE_ℓ's, MC_m's, and $1\lambda_o$). The implicit function theorem[1] is then applied to these equations yielding the following equations:

$$\hat{E}_\ell = E_\ell(r_1, \ldots, r_o) \qquad \ell = 1, \ldots, L$$

$$\hat{C}_m = C_m(r_1, \ldots, r_o) \qquad m = 1, \ldots, M$$

$$\hat{\lambda}_o = \lambda_o(r_1, \ldots, r_o)$$

where the symbol $\hat{}$ designates the optimal value. The next stage of analysis considers what information is available concerning the slopes of these equations. Specifically, the nature of $\frac{\partial \hat{E}_\ell}{\partial r_k}$ and $\frac{\partial \hat{C}_m}{\partial r_k}$ will be studied.

The slope analysis proceeds exactly as in traditional consumer theory. First, the necessary conditions equations are differentiated with respect to r_k. This yields the set of linear equations shown on the next page.[2] If the matrix of the set of linear equations is denoted by D, then the solution of the equations takes the form

$$\frac{\partial E_\ell}{\partial r_k} = (|D|)^{-1} \sum_{\ell=1}^{L} \left(\lambda_o \frac{\partial^2 H}{\partial r_k \partial E_\ell} D_{m\ell}\right) + \sum_{m=1}^{M} \left(\lambda_o \frac{\partial^2 H}{\partial r_k \partial C_m} D_{m\ell}\right) + \frac{\partial H}{\partial r_k} D_{L+M+1,\ell}$$

where $D_{m\ell}$ is the cofactor of the m^{th} row and ℓ^{th} column.

To understand the nature of this equation for the slope, the following minimization problem is considered. This problem,

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[1]The reader may check to verify that the hypothese of the theorem are satisfied in this case.

[2]This form of D is obtained by differentiating the equations as given and then multiplying the last $m + 1$ equations by minus one.

$$
\left[\begin{array}{ccc|ccc|c}
1 & \cdots\ \ell\ \cdots & L & 1 & \cdots\ m\ \cdots & M & -H_{E_1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
\ell & \longleftarrow (U_{E_\ell E_\ell} - \lambda_0 H_{E_\ell E_\ell}) \longrightarrow & & \ell & \longleftarrow (U_{C_m E_\ell} - \lambda_0 H_{C_m E_\ell}) \longrightarrow & & -H_{E_\ell} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
L & \cdots & \cdots & L & \cdots & \cdots & -H_{E_L} \\
\hline
1 & \cdots\ \ell\ \cdots & L & 1 & \cdots\ m\ \cdots & M & H_{C_1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
m & \longleftarrow (U_{E_\ell C_m} - \lambda_0 H_{E_\ell C_m}) \longrightarrow & & m & \longleftarrow (U_{C_m C_m} - \lambda_0 H_{C_m C_m}) \longrightarrow & & H_{C_m} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
M & \cdots & \cdots & M & \cdots & \cdots & H_{C_M} \\
\hline
-H_{E_1} & \cdots\ -H_{E_\ell}\ \cdots & -H_{E_L} & H_{C_1} & \cdots\ H_{C_m}\ \cdots & H_{C_M} & 0
\end{array}\right]
\left[\begin{array}{c}
\frac{\partial \hat{E}_1}{\partial r_k} \\ \vdots \\ \frac{\partial \hat{E}_\ell}{\partial r_k} \\ \vdots \\ \frac{\partial \hat{E}_L}{\partial r_k} \\
\hline
\frac{\partial \hat{C}_1}{\partial r_k} \\ \vdots \\ \frac{\partial \hat{C}_m}{\partial r_k} \\ \vdots \\ \frac{\partial \hat{C}_M}{\partial r_k} \\
\hline
\frac{\partial \hat{\lambda}_0}{\partial r_k}
\end{array}\right]
=
\left[\begin{array}{c}
\lambda_0 H_{r_k E_1} \\ \vdots \\ \lambda_0 H_{r_k E_\ell} \\ \vdots \\ \lambda_0 H_{r_k E_L} \\
\hline
\lambda_0 H_{r_k C_1} \\ \vdots \\ \lambda_0 H_{r_k C_m} \\ \vdots \\ \lambda_0 H_{r_k C_M} \\
\hline
H_{r_k}
\end{array}\right]
$$

$$\min_{\underline{E},\ \underline{C}} H(\underline{E}, -\underline{C}, \underline{r})$$

s.t.

$$U(\underline{E}, -\underline{C}) = U^*$$

$$E_\ell,\ C_m > 0 \quad \text{for all} \quad \ell, m$$

has the Lagrangian function:

$$\psi(\underline{E}, -\underline{C}, \mu_o) = H(\underline{E}, -\underline{C}, \underline{r}) - \mu_o[U(\underline{E}, -\underline{C}) - U^*]$$

Applying the Kuhn-Tucker theorem[1] the necessary conditions are

$$(9)\quad \frac{\partial H}{\partial E_\ell} - \mu_o \frac{\partial U}{\partial E_\ell} = 0 \qquad j = 1, \ldots, n$$

$$(10)\quad -\frac{\partial H}{\partial c_m} + \mu_o \frac{\partial U}{\partial c_m} = 0 \qquad m = 1, \ldots, M$$

$$(11)\quad U(\underline{E}, -\underline{C}) - U^* = 0$$

The reader may quickly check that the decision rules derived from these equations are the same as those already discussed. In addition, $\mu_o = \lambda_o^{-1}$.

Differentiating equation sets (9), (10), and (11) yields the set of linear equations exhibited on the next page.[2] Denoting the matrix of this set of equations by G, it can be shown that

$$|D| = \lambda_o^{-2}|G|.$$

Using this information, the first two terms of the equation $\frac{\partial \hat{E}_\ell}{\partial r_k}$ can easily be shown to be $\left.\frac{\partial \hat{E}}{\partial r_k}\right|_U$. That is, the first two terms are the

---

[1]The reader may verify that the hypotheses are satisfied.

[2]This form is obtained by differentiating the equations as given and multiplying the first L and the last one by minus one.

$$
\begin{array}{c}
\left[\begin{array}{ccccc|ccccc|c}
1 & \cdots & \ell & \cdots & L & 1 & \cdots & m & \cdots & M & U_{E_1} \\
\vdots & & \uparrow & & \vdots & \vdots & & \uparrow & & \vdots & \vdots \\
\ell & \leftarrow & (-H_{E_\ell E_\ell} + \mu_0 U_{E_\ell E_\ell}) & \rightarrow & & \ell & \leftarrow & (-H_{C_m E_\ell} + \mu_0 U_{C_m E_\ell}) & \rightarrow & & U_{E_\ell} \\
\vdots & & \downarrow & & \vdots & \vdots & & \downarrow & & \vdots & \vdots \\
L & \cdots & & \cdots & & L & \cdots & & \cdots & & U_{E_L} \\
\hline
1 & \cdots & \ell & \cdots & L & 1 & \cdots & m & \cdots & M & U_{C_1} \\
\vdots & & \uparrow & & \vdots & \vdots & & \uparrow & & \vdots & \vdots \\
m & \leftarrow & (-H_{E_\ell C_m} + \mu_0 U_{E_\ell C_m}) & \rightarrow & & m & \leftarrow & (-H_{C_m C_m} + \mu_0 U_{C_m C_m}) & \rightarrow & & U_{C_m} \\
\vdots & & \downarrow & & \vdots & \vdots & & \downarrow & & \vdots & \vdots \\
M & \cdots & & \cdots & & M & \cdots & & \cdots & & U_{C_M} \\
\hline
-U_{E_1} & \cdots & -U_{E_\ell} & \cdots & -U_{E_L} & U_{C_1} & \cdots & U_{C_m} & \cdots & U_{C_M} & 0
\end{array}\right]
\left[\begin{array}{c}
\frac{\partial \hat{E}_1}{\partial r_k} \\ \vdots \\ \frac{\partial \hat{E}_\ell}{\partial r_k} \\ \vdots \\ \frac{\partial \hat{E}_L}{\partial r_k} \\ \hline \frac{\partial \hat{C}_1}{\partial r_1} \\ \vdots \\ \frac{\partial \hat{C}_m}{\partial r_k} \\ \vdots \\ \frac{\partial \hat{C}_M}{\partial r_k} \\ \hline \frac{\partial \hat{C}_0}{\partial r_k}
\end{array}\right]
=
\left[\begin{array}{c}
H_{r_k E_1} \\ \vdots \\ H_{r_k E_\ell} \\ \vdots \\ H_{r_k E_L} \\ \hline H_{r_k C_1} \\ \vdots \\ H_{r_k C_m} \\ \vdots \\ H_{r_k C_M} \\ \hline 0
\end{array}\right]
\end{array}
$$

change induced in the optimal value of the ℓ^{th} benefit measure by a change in the k^{th} parameter holding utility constant. In traditional consumer theory, this is known as the substitution effect and it will be so named here.

In the original maximization problem, the cost-benefit production function being of implicit form is equal to zero. However, at this stage of the analysis a shift of the entire production surface is to be studied. This is most easily done by setting the implicit production function equal to a parameter z and varying it, holding all the r_k's constant. As would be expected, this changes the nature of the linear system of equations on page 23 not at all. The variables are now partial derivatives with respect to z. Finally, the right side of the equation set is a column vector of zeros except for the last element which is 1. Hence, $D_{L+M+1,\ell}$ can be identified as $\left.\frac{\partial E_\ell}{\partial z}\right|_{r's}$. That is, this is the effect of a shift of the production surface holding the r_k's constant. By removing $\left(\frac{\partial H}{\partial E_\ell}\right)^{-1}$ from $|D_{L+M+1,\ell}|$, the slope equation can be written

$$\frac{\partial E_\ell}{\partial r_k} = \left.\frac{\partial E_\ell}{\partial r_k}\right|_U + \frac{\frac{\partial H}{\partial r_k}}{\frac{\partial H}{\partial E_\ell}} \cdot \left.\frac{\partial E_\ell}{\partial z}\right|_{r's} .$$

Now,

$$-\frac{\partial E_\ell}{\partial r_k} = \frac{\frac{\partial H}{\partial r_k}}{\frac{\partial H}{\partial E_\ell}} .$$

This is the direct production effect on E_ℓ due to a change in r_k, and will be called the rate of benefit measure ℓ productivity with

respect to a change in parameter $r_k(RE_\ell r_k P)$. Hence, the slope of the demand function can be written as

$$\frac{\partial E_\ell}{\partial r_k} = \left.\frac{\partial E_\ell}{\partial r_k}\right|_U - (RE_\ell r_k P) \left.\frac{\partial E_\ell}{\partial z}\right|_{r's}$$

In the analogous manner to consumer theory, the latter term will be called the income effect.

Thus, when one of the production function shift parameters changes, the change in demand for a benefit can be resolved into two parts. The first will be called the private substitution effect, and the latter, the private income effect. A similar relationship occurs with cost measures.

As in traditional consumer theory, private substitutes, compliments, superior and inferior "goods" may be defined. The definitions given below are extensions of the familiar consumer definitions.

<u>Private Substitutes</u>

Two benefit or two cost or one benefit-one cost measure will be called private substitutes with respect to parameter r_k if

$$\left.\frac{\partial E_\ell}{\partial r_k}\right|_U > 0 \quad \text{and} \quad \left.\frac{\partial E_\alpha}{\partial r_k}\right|_U < 0 \qquad \alpha \neq \ell$$

$$\left.\frac{\partial C_m}{\partial r_k}\right|_U > 0 \quad \text{and} \quad \left.\frac{\partial C_\alpha}{\partial r_k}\right|_U < 0 \qquad k \neq m$$

$$\left.\frac{\partial E_\ell}{\partial r_k}\right|_U > 0 \quad \text{and} \quad \left.\frac{\partial C_m}{\partial r_k}\right|_U < 0$$

$$\left.\frac{\partial C_m}{\partial r_k}\right|_U > 0 \quad \text{and} \quad \left.\frac{\partial E_\ell}{\partial r_k}\right|_U < 0$$

Private Complements

Two benefit or two cost or one benefit-one cost measure(s) will be called private complements if the negative signs in the private substitute definition are positive and the positive signs remain unchanged.

Private Superior Measures

A benefit or cost measure will be called a private superior measure if

$$\left.\frac{\partial E_\ell}{\partial z}\right|_{r's} > 0 \quad \text{or} \quad \left.\frac{\partial C_m}{\partial z}\right|_{r's} > 0$$

Private Inferior Measures

A benefit or cost measure will be called a private inferior measure if

$$\left.\frac{\partial E_\ell}{\partial z}\right|_{r's} < 0 \quad \text{or} \quad \left.\frac{\partial C_m}{\partial z}\right|_{r's} < 0$$

In consumer theory these are called the weak definitions. The gross definitions can also be defined and are left to the reader. It is emphasized that this is a set of definitions referring to the private -- subjective -- decisions of the decision-maker.

Because there are so many cases of constraints, slack or binding, and necessary conditions ((1) and (2)) being strict inequalities or equations, no catalogue of results will be given. Cases of interest can be considered by the reader. The bindingness or slackness of constraints can be handled using the theorems in King [7]. Strict inequality necessary conditions can be converted to equalities by use of a slack variable. Care must be exercised, however, when some cost

equals its budget level, and the indifference surface and production surface are tangent at that point. Here the derivative does not exist where a cost measure is considered with respect to a change in its own budget level. One-sided derivatives (to the optimum point) can, however, give information. More generally, this multiplicity of cases yields no sign of slope information that appears useful. Finally, note that some slopes may be zero, and hence the substitute/compliment and superior/inferior definitions are not usable. One could extend these concepts to include a zero slope, or possibly preferably consider defining a new category.

SECTION VII

CONCLUSIONS

In this paper a mathematical model of a governmental decision-maker's problem of choice from a set of alternative cost-benefit streams is presented. The choice objects, costs, and benefits are considered to be defined physically/socially, in time, in space, and by a state-of-nature. The decision-maker's preferences with respect to costs and benefits are represented by a utility index of the standard type except that costs are assumed to be a disutility-causing entity. The set of alternatives from which choice is described is represented by a cost-benefit surface. The decision problem is formulated as maximization problems. Decision rules are derived. Comparatibe statics results are given.

While a general framework for understanding decision-making in this context is presented, it must be noted that this does not complete the research needed in this area. There is opportunity for use of the Qualitative Calculus. In addition, this model is an equilibrium model, and research is needed in the dynamics of the phenomenon.

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13. ABSTRACT

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